GLOBAL EDITION

THOMAS' CALCULUS

Early Transcendentals

Fourteenth Edition in SI Units

Hass • Heil • Weir

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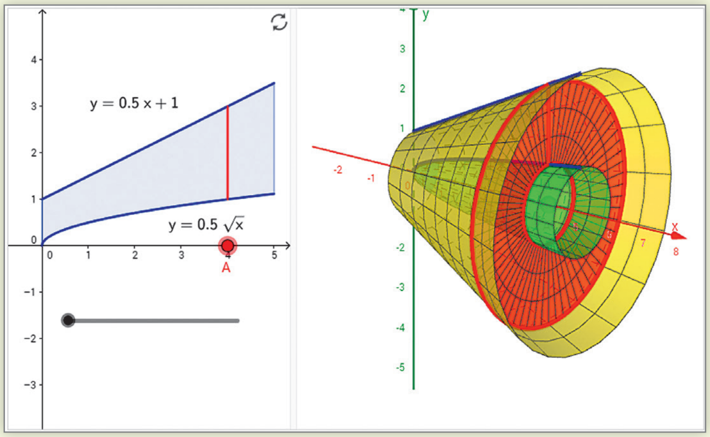

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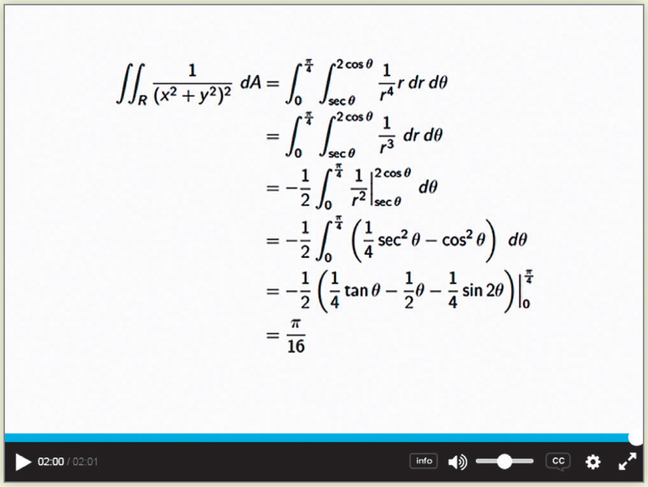
Interactive Figures illustrate key concepts and allow manipulation for use as teaching and learning tools. These figures help encourage active learning, critical thinking, and conceptual understanding. We also include videos that use the Interactive Figures to explain key concepts.

Instructional Videos

MyLab Math contains videos to help students be successful in calculus. The tutorial videos cover key concepts from the text and are especially handy for use in a flipped classroom or when a student misses a lecture.



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THOMAS'

CALCULUS

Early Transcendentals

FOURTEENTH EDITION IN SI UNITS

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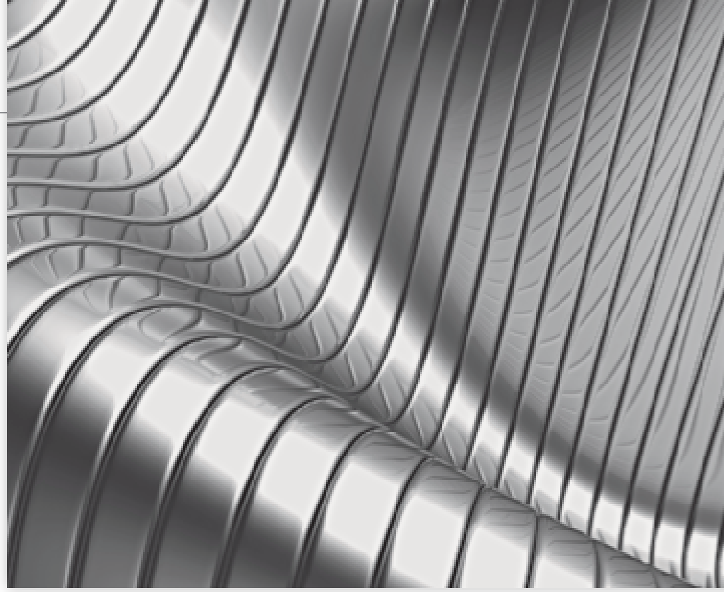


Preface

Thomas' Calculus: Early Transcendentals, Fourteenth Edition in SI Units, provides a modern introduction to calculus that focuses on developing conceptual understanding of the underlying mathematical ideas. This text supports a calculus sequence typically taken by students in STEM fields over several semesters. Intuitive and precise explanations, thoughtfully chosen examples, superior figures, and time-tested exercise sets are the foundation of this text. We continue to improve this text in keeping with shifts in both the preparation and the goals of today's students, and in the applications of calculus to a changing world.

Many of today's students have been exposed to calculus in high school. For some, this translates into a successful experience with calculus in college. For others, however, the result is an overconfidence in their computational abilities coupled with underlying gaps in algebra and trigonometry mastery, as well as poor conceptual understanding. In this text, we seek to meet the needs of the increasingly varied population in the calculus sequence. We have taken care to provide enough review material (in the text and appendices), detailed solutions, and a variety of examples and exercises, to support a complete understanding of calculus for students at varying levels. Within the text, we present the material in a way that supports the development of mathematical maturity, going beyond memorizing formulas and routine procedures, and we show students how to generalize key concepts once they are introduced. References are made throughout, tying new concepts to related ones that were studied earlier. After studying calculus from *Thomas*, students will have developed problem-solving and reasoning abilities that will serve them well in many important aspects of their lives. Mastering this beautiful and creative subject, with its many practical applications across so many fields, is its own reward. But the real gifts of studying calculus are acquiring the ability to think logically and precisely; understanding what is defined, what is assumed, and what is deduced; and learning how to generalize conceptually. We intend this book to encourage and support those goals.

New to This Edition

We welcome to this edition a new coauthor, Christopher Heil from the Georgia Institute of Technology. He has been involved in teaching calculus, linear algebra, analysis, and abstract algebra at Georgia Tech since 1993. He is an experienced author and served as a consultant on the previous edition of this text. His research is in harmonic analysis, including time-frequency analysis, wavelets, and operator theory.

This is a substantial revision. Every word, symbol, and figure was revisited to ensure clarity, consistency, and conciseness. Additionally, we made the following text-wide updates:

- Updated graphics to bring out clear visualization and mathematical correctness.
- Added examples (in response to user feedback) to overcome conceptual obstacles. See Example 3 in Section 16.1.
- Added new types of homework exercises throughout, including many with a geometric nature. The new exercises are not just more of the same, but rather give different perspectives on and approaches to each topic. We also analyzed aggregated student usage and performance data from MyLab Math for the previous edition of this text. The results of this analysis helped improve the quality and quantity of the exercises.
- Added short URLs to historical links that allow students to navigate directly to online information.
- Added new marginal notes throughout to guide the reader through the process of problem solution and to emphasize that each step in a mathematical argument is rigorously justified.

New to MyLab Math

Many improvements have been made to the overall functionality of MyLab Math since the previous edition. Beyond that, we have also increased and improved the content specific to this text.

- Instructors now have more exercises than ever to choose from in assigning homework.
- The MyLab Math exercise-scoring engine has been updated to allow for more robust coverage of certain topics, including differential equations.
- A full suite of Interactive Figures have been added to support teaching and learning. The figures are designed to be used in lecture, as well as by students independently. The figures are editable using the freely available GeoGebra software. The figures were created by Marc Renault (Shippensburg University), Kevin Hopkins (Southwest Baptist University), Steve Phelps (University of Cincinnati), and Tim Brzezinski (Berlin High School, CT).
- Enhanced Sample Assignments include just-in-time prerequisite review, help keep skills fresh with distributed practice of key concepts (based on research by Jeff Hieb, Keith Lyle, and Pat Ralston of University of Louisville), and provide opportunities to work exercises without learning aids (to help students develop confidence in their ability to solve problems independently).
- Additional Conceptual Questions augment text exercises to focus on deeper, theoretical understanding of the key concepts in calculus. These questions were written by faculty at Cornell University under an NSF grant. They are also assignable through Learning Catalytics.
- Setup & Solve exercises now appear in many sections. These exercises require students to show how they set up a problem as well as the solution, better mirroring what is required of students on tests.
- New instructional videos by Greg Wisloski and Dan Radelet (both of Indiana University of PA) augment the already robust collection within the course. These videos support the overall approach of the text—specifically, they go beyond routine procedures to show students how to generalize and connect key concepts.

Content Enhancements

Chapter 1

- Clarified explanation of definition of exponential function in 1.4.
- Replaced $\sin^{-1}$ notation for the inverse sine function with arcsin as default notation in 1.5, and similarly for other trig functions.
- Added new Exercises: **1.1:** 59–62, **1.2:** 21–22; **1.3:** 64–65, **1.5:** 61–64, 79cd; **PE:** 29–32.

Chapter 2

- Added definition of average speed in 2.1.
- Updated definition of limits to allow for arbitrary domains. The definition of limits is now consistent with the definition in multivariable domains later in the text and with more general mathematical usage.
- Reworded limit and continuity definitions to remove implication symbols and improve comprehension.
- Added new Example 7 in 2.4 to illustrate limits of ratios of trig functions.
- Rewrote 2.6 Example 11 to solve the equation by finding a zero, consistent with previous discussion.
- Added new Exercises: **2.1:** 15–18; **2.2:** 3h–k, 4f–I; **2.4:** 19–20, 45–46; **2.5:** 69–74; **2.6:** 31–32; **PE:** 57–58; **AAE:** 35–38.

Chapter 3

- Clarified relation of slope and rate of change.
- Added new Figure 3.9 using the square root function to illustrate vertical tangent lines.
- Added figure of $x \sin(1/x)$ in 3.2 to illustrate how oscillation can lead to non-existence of a derivative of a continuous function.
- Revised product rule to make order of factors consistent throughout text, including later dot product and cross product formulas.
- Added new Exercises: **3.2:** 36, 43–44; **3.3:** 65–66; **3.5:** 43–44, 61bc; **3.6:** 79–80, 111–113; **3.7:** 27–28; **3.8:** 97–100; **3.9:** 43–46; **3.10:** 47; **AAE:** 14–15, 26–27.

Chapter 4

- Added summary to 4.1.
- Added new Example 12 with new Figure 4.35 to give basic and advanced examples of concavity.
- Added new Exercises: **4.1:** 53–56, 67–70; **4.3:** 45–46, 67–68; **4.4:** 107–112; **4.6:** 37–42; **4.7:** 7–10; **4.8:** 115–118; **PE:** 1–16, 101–102; **AAE:** 19–20, 38–39. Moved Exercises 4.1: 53–68 to PE.

Chapter 5

- Improved discussion in 5.4 and added new Figure 5.18 to illustrate the Mean Value Theorem.
- Added new Exercises: **5.2:** 33–36; **5.4:** 71–72; **5.6:** 47–48; **PE:** 43–44, 75–76.

Chapter 6

- Clarified cylindrical shell method.
- Added introductory discussion of mass distribution along a line, with figure, in 6.6.
- Added new Exercises: **6.1:** 15; **6.2:** 49–50; **6.3:** 13–14; **6.5:** 1–2; **6.6:** 1–6, 21–22; **PE:** 17–18, 23–24, 37–38.

Chapter 7

- Clarified discussion of separable differential equations in 7.2.
- Added new Exercises: **7.1:** 61–62, 73; **PE:** 41–42.

Chapter 8

- Updated 8.2 Integration by Parts discussion to emphasize $u(x)v'(x)\,dx$ form rather than $u\,dv$. Rewrote Examples 1–3 accordingly.
- Removed discussion of tabular integration and associated exercises.
- Updated discussion in 8.5 on how to find constants in Partial Fraction method.
- Updated notation in 8.8 to align with standard usage in statistics.
- Added new Exercises: **8.1:** 41–44; **8.2:** 53–56, 72–73; **8.3:** 75–76; **8.4:** 49–52; **8.5:** 51–66, 73–74; **8.8:** 35–38, 77–78; **PE:** 69–88.

Chapter 9

- Clarified the different meaning of a sequence and a series.
- Added new Figure 9.9 to illustrate sum of a series as area of a histogram.
- Added to 9.3 a discussion on the importance of bounding errors in approximations.
- Added new Figure 9.13 illustrating how to use integrals to bound remainder terms of partial sums.
- Rewrote Theorem 10 in 9.4 to bring out similarity to the integral comparison test.
- Added new Figure 9.16 to illustrate the differing behaviors of the harmonic and alternating harmonic series.
- Renamed the nth term test the "nth term test for divergence" to emphasize that it says nothing about convergence.
- Added new Figure 9.19 to illustrate polynomials converging to $\ln(1 + x)$, which illustrates convergence on the half-open interval $(-1, 1]$.
- Used red dots and intervals to indicate intervals and points where divergence occurs and blue to indicate convergence throughout Chapter 9.
- Added new Figure 9.21 to show the six different possibilities for an interval of convergence.
- Added new Exercises: **9.1:** 27–30, 72–77; **9.2:** 19–22, 73–76, 105; **9.3:** 11–12, 39–42; **9.4:** 55–56; **9.5:** 45–46, 65–66; **9.6:** 57–82; **9.7:** 61–65; **9.8:** 23–24, 39–40; **9.9:** 11–12, 37–38; **PE:** 41–44, 97–102.

Chapter 10

- Added new Example 1 and Figure 10.2 in 10.1 to give a straightforward first example of a parametrized curve.
- Updated area formulas for polar coordinates to include conditions for positive r and non-overlapping θ.
- Added new Example 3 and Figure 10.37 in 10.4 to illustrate intersections of polar curves.
- Added new Exercises: **10.1:** 19–28; **10.2:** 49–50; **10.4:** 21–24.

Chapter 11

- Added new Figure 11.13(b) to show the effect of scaling a vector.
- Added new Example 7 and Figure 11.26 in 11.3 to illustrate projection of a vector.
- Added discussion on general quadric surfaces in 11.6, with new Example 4 and new Figure 11.48 illustrating the description of an ellipsoid not centered at the origin via completing the square.
- Added new Exercises: **11.1:** 31–34, 59–60, 73–76; **11.2:** 43–44; **11.3:** 17–18; **11.4:** 51–57; **11.5:** 49–52.

Chapter 12

- Added sidebars on how to pronounce Greek letters such as kappa, tau, etc.
- Added new Exercises: **12.1:** 1–4, 27–36; **12.2:** 15–16, 19–20; **12.4:** 27–28; **12.6:** 1–2.

Chapter 13

- Elaborated on discussion of open and closed regions in 13.1.
- Standardized notation for evaluating partial derivatives, gradients, and directional derivatives at a point, throughout the chapter.
- Renamed "branch diagrams" as "dependency diagrams" which clarifies that they capture dependence of variables.
- Added new Exercises: **13.2:** 51–54; **13.3:** 51–54, 59–60, 71–74, 103–104; **13.4:** 20–30, 43–46, 57–58; **13.5:** 41–44; **13.6:** 9–10, 61; **13.7:** 61–62.

Chapter 14

- Added new Figure 14.21b to illustrate setting up limits of a double integral.
- Added new 14.5 Example 1, modified Examples 2 and 3, and added new Figures 14.31, 14.32, and 14.33 to give basic examples of setting up limits of integration for a triple integral.
- Added new material on joint probability distributions as an application of multivariable integration.
- Added new Examples 5, 6 and 7 to Section 14.6.
- Added new Exercises: **14.1:** 15–16, 27–28; **14.6:** 39–44; **14.7:** 1–22.

Chapter 15

- Added new Figure 15.4 to illustrate a line integral of a function.
- Added new Figure 15.17 to illustrate a gradient field.
- Added new Figure 15.19 to illustrate a line integral of a vector field.
- Clarified notation for line integrals in 15.2.
- Added discussion of the sign of potential energy in 15.3.
- Rewrote solution of Example 3 in 15.4 to clarify connection to Green's Theorem.
- Updated discussion of surface orientation in 15.6 along with Figure 15.52.
- Added new Exercises: **15.2:** 37–38, 41–46; **15.4:** 1–6; **15.6:** 49–50; **15.7:** 1–6; **15.8:** 1–4.

Chapter 16

- Added new Example 3 with Figure 16.3 to illustrate how to construct a slope field.
- Added new Exercises: **16.1:** 11–14; **PE:** 17–22, 43–44.

Appendices: Rewrote Appendix 8 on complex numbers. Shortened Appendix 2 to focus on issues arising in use of mathematical software and potential pitfalls.

Continuing Features

Rigor The level of rigor is consistent with that of earlier editions. We continue to distinguish between formal and informal discussions and to point out their differences. Starting with a more intuitive, less formal approach helps students understand a new or difficult concept so they can then appreciate its full mathematical precision and outcomes. We pay attention to defining ideas carefully and to proving theorems appropriate for calculus students, while mentioning deeper or subtler issues they would study in a more advanced course. Our organization and distinctions between informal and formal discussions give the instructor a degree of flexibility in the amount and depth of coverage of the various topics. For example, while we do not prove the Intermediate Value Theorem or the Extreme Value Theorem for continuous functions on a closed finite interval, we do state these theorems precisely, illustrate their meanings in numerous examples, and use them to prove other important results. Furthermore, for those instructors who desire greater depth of coverage, in Appendix 7 we discuss the reliance of these theorems on the completeness of the real numbers.

Writing Exercises Writing exercises placed throughout the text ask students to explore and explain a variety of calculus concepts and applications. In addition, the end of each chapter contains a list of questions for students to review and summarize what they have learned. Many of these exercises make good writing assignments.

End-of-Chapter Reviews and Projects In addition to problems appearing after each section, each chapter culminates with review questions, practice exercises covering the entire chapter, and a series of Additional and Advanced Exercises with more challenging or synthesizing problems. Most chapters also include descriptions of several **Technology Application Projects** that can be worked by individual students or groups of students over a longer period of time. These projects require the use of *Mathematica* or *Maple*, along with pre-made files that are available for download within MyLab Math.

Writing and Applications This text continues to be easy to read, conversational, and mathematically rich. Each new topic is motivated by clear, easy-to-understand examples and is then reinforced by its application to real-world problems of immediate interest to students. A hallmark of this book has been the application of calculus to science and engineering. These applied problems have been updated, improved, and extended continually over the last several editions.

Technology In a course using the text, technology can be incorporated according to the taste of the instructor. Each section contains exercises requiring the use of technology; these are marked with a T if suitable for calculator or computer use, or they are labeled **Computer Explorations** if a computer algebra system (CAS, such as *Maple* or *Mathematica*) is required.

Additional Resources

MyLab Math® Online Course (access code required)

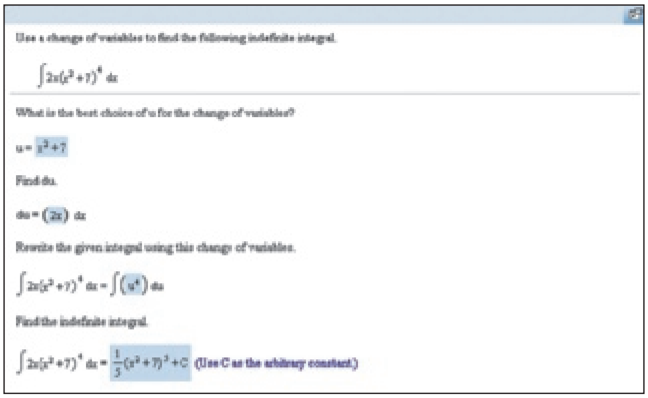
Built around Pearson's best-selling content, MyLab Math is an online homework, tutorial, and assessment program designed to work with this text to engage students and improve results. MyLab Math can be successfully implemented in any classroom environment—lab-based, hybrid, fully online, or traditional.

Used by more than 37 million students worldwide, MyLab Math delivers consistent, measurable gains in student learning outcomes, retention, and subsequent course success. Visit **www.mymathlab.com/results** to learn more.

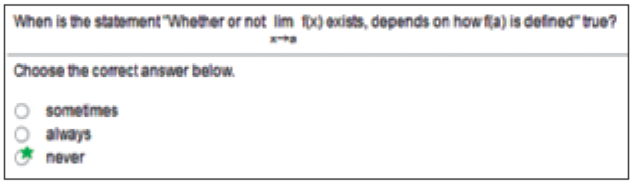
Motivation Students are motivated to succeed when they're engaged in the learning experience and understand the relevance and power of mathematics. MyLab Math's online homework offers students immediate feedback and tutorial assistance that motivates them to do more, which means they retain more knowledge and improve their test scores.

- **Exercises with immediate feedback**—assignable exercises for this text regenerate algorithmically to give students unlimited opportunity for practice and mastery. MyLab Math provides helpful feedback when students enter incorrect answers and includes optional learning aids such as Help Me Solve This, View an Example, videos, and an eText.

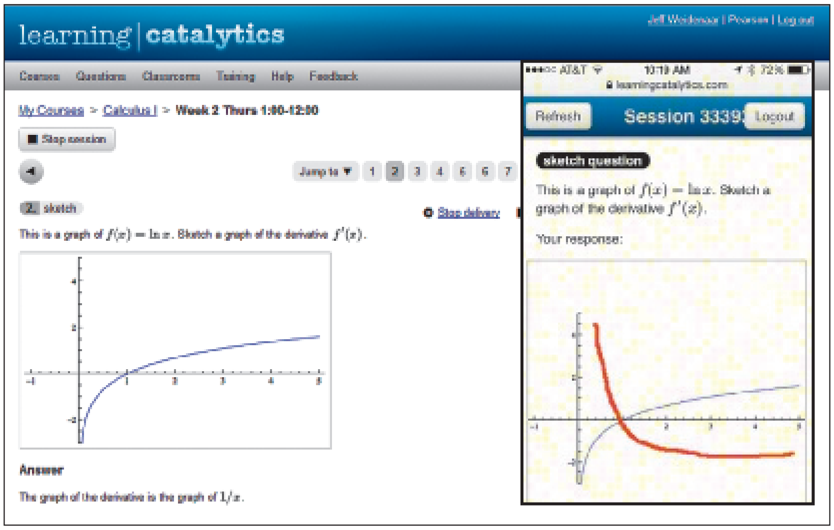
- **Setup and Solve Exercises** ask students to first describe how they will set up and approach the problem. This reinforces students' conceptual understanding of the process they are applying and promotes long-term retention of the skill.



- **Additional Conceptual Questions** focus on deeper, theoretical understanding of the key concepts in calculus. These questions were written by faculty at Cornell University under an NSF grant and are also assignable through Learning Catalytics.

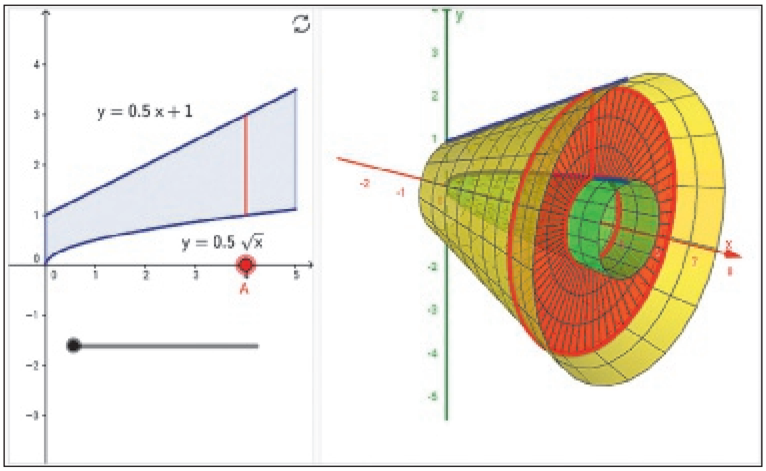


- **Learning Catalytics™** is a student response tool that uses students' smartphones, tablets, or laptops to engage them in more interactive tasks and thinking during lecture. Learning Catalytics fosters student engagement and peer-to-peer learning with real-time analytics. Learning Catalytics is available to all MyLab Math users.

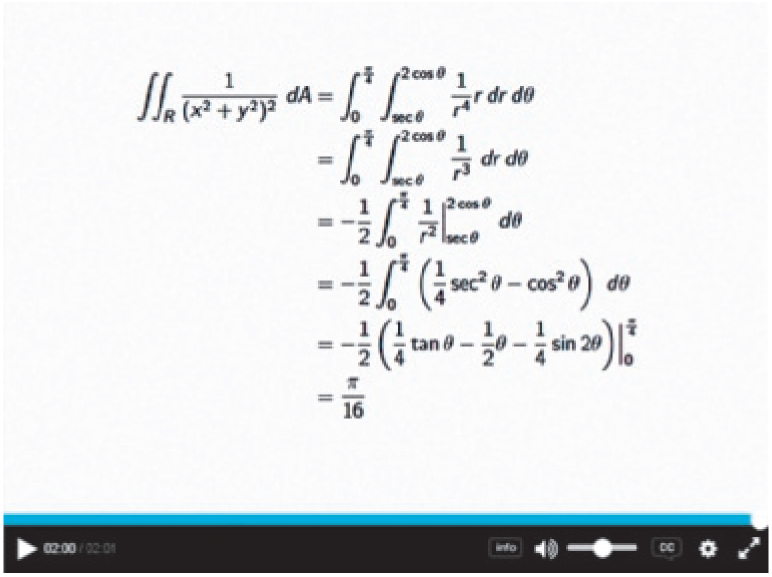


Learning and Teaching Tools

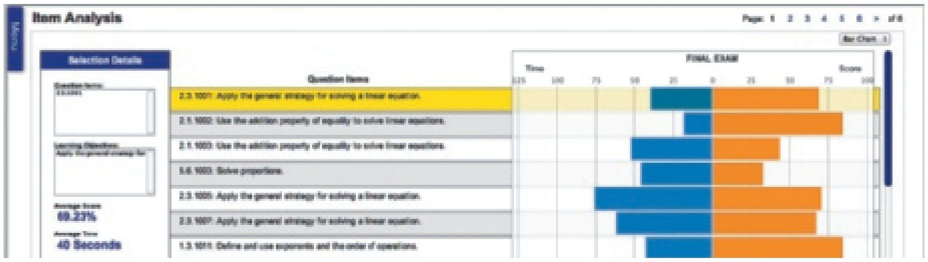
- **Interactive Figures** bring calculus concepts to life, helping you understand key ideas by working with their visual representations. They illustrate key concepts and allow manipulation for use as teaching and learning tools. We also include videos that use the Interactive Figures to explain key concepts.



- **Instructional videos**—Hundreds of videos are available as learning aids within exercises and for self-study. The tutorial videos cover key concepts from your text and are especially handy if you miss a lecture or just need another explanation. The Guide to Video-Based Assignments makes it easy to assign videos for homework by showing which MyLab Math exercises correspond to each video.



- **The complete eText** is available to students through their MyLab Math courses for the lifetime of the edition, giving students unlimited access to the eText within any course using that edition of the text.
- **Enhanced Sample Assignments** These assignments include just-in-time prerequisite review, help keep skills fresh with distributed practice of key concepts, and provide opportunities to work exercises without learning aids so students can check their understanding.
- **PowerPoint Presentations** that cover each section of the book are available for download.
- **Mathematica manual and projects, Maple manual and projects, TI Graphing Calculator manual**—These manuals cover *Maple* 17, *Mathematica 8,* and the TI-84 Plus and TI-89, respectively. Each provides detailed guidance for integrating the software package or graphing calculator throughout the course, including syntax and commands.
- **Accessibility** and achievement go hand in hand. MyLab Math is compatible with the JAWS screen reader, and it enables students to read and interact with multiple-choice and free-response problem types via keyboard controls and math notation input. MyLab Math also works with screen enlargers, including ZoomText, MAGic, and SuperNova. And, all MyLab Math videos have closed-captioning. More information is available at **http://mymathlab.com/accessibility**.
- **A comprehensive gradebook** with enhanced reporting functionality allows you to efficiently manage your course.
  - **The Reporting Dashboard** offers insight as you view, analyze, and report learning outcomes. Student performance data is presented at the class, section, and program levels in an accessible, visual manner so you'll have the information you need to keep your students on track.



- **Item Analysis** tracks class-wide understanding of particular exercises so you can refine your class lectures or adjust the course/department syllabus. Just-in-time teaching has never been easier!

MyLab Math comes from an experienced partner with educational expertise and an eye on the future. Whether you are just getting started with MyLab Math, or have a question along the way, we're here to help you learn about our technologies and how to incorporate them into your course. To learn more about how MyLab Math helps students succeed, visit **www.mymathlab.com** or contact your Pearson rep.

Instructor's Solutions Manual (downloadable)

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The Instructor's Solutions Manual contains complete worked-out solutions to all the exercises in *Thomas' Calculus: Early Transcendentals*. It can be downloaded from within MyLab Math or the Pearson Instructor Resource Center, **www.pearsonglobaleditions.com.**

Student's Solutions Manual

ISBN: 1-292-25319-3 | 978-1-292-25319-0

The Student's Solutions Manual contains worked-out solutions to all the odd-numbered exercises in *Thomas' Calculus: Early Transcendentals*. These manuals can be downloaded from within MyLab Math.

Companion Website

The companion Website, located at **www.pearsonglobaleditions.com**, includes opportunities for practice and review.

Just-In-Time Algebra and Trigonometry for Early Transcendentals Calculus, Fourth Edition

ISBN 0-321-67103-1 | 978-0-321-67103-5

Sharp algebra and trigonometry skills are critical to mastering calculus, and *Just-in-Time Algebra and Trigonometry for Early Transcendentals Calculus* by Guntram Mueller and Ronald I. Brent is designed to bolster these skills while students study calculus. As students make their way through calculus, this brief supplementary text is with them every step of the way, showing them the necessary algebra or trigonometry topics and pointing out potential problem spots. The easy-to-use table of contents has topics arranged in the order in which students will need them as they study calculus.

Technology Manuals and Projects (downloadable)

Maple Manual and Projects by Marie Vanisko, Carroll College
Mathematica Manual and Projects by Marie Vanisko, Carroll College
TI-Graphing Calculator Manual by Elaine McDonald-Newman, Sonoma State University

These manuals and projects cover Maple 17, Mathematica 9, and the TI-84 Plus and TI-89. Each manual provides detailed guidance for integrating a specific software package or graphing calculator throughout the course, including syntax and commands. The projects include instructions and ready-made application files for Maple and Mathematica. These materials are available to download within MyLab Math.

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These classroom presentation slides were created for the *Thomas' Calculus* series. Key graphics from the book are included to help bring the concepts alive in the classroom. These files are available to qualified instructors through the Pearson Instructor Resource Center, **www.pearsonglobaleditions.com**, and within MyLab Math.

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Dedication

We regret that prior to the writing of this edition our coauthor Maurice Weir passed away. Maury was dedicated to achieving the highest possible standards in the presentation of mathematics. He insisted on clarity, rigor, and readability. Maury was a role model to his students, his colleagues, and his coauthors. He was very proud of his daughters, Maia Coyle and Renee Waina, and of his grandsons, Matthew Ryan and Andrew Dean Waina. He will be greatly missed.

1

Functions

OVERVIEW Functions are fundamental to the study of calculus. In this chapter we review what functions are and how they are visualized as graphs, how they are combined and transformed, and ways they can be classified.

1.1 Functions and Their Graphs

Functions are a tool for describing the real world in mathematical terms. A function can be represented by an equation, a graph, a numerical table, or a verbal description; we will use all four representations throughout this book. This section reviews these ideas.

Functions; Domain and Range

The temperature at which water boils depends on the elevation above sea level. The interest paid on a cash investment depends on the length of time the investment is held. The area of a circle depends on the radius of the circle. The distance an object travels depends on the elapsed time.

In each case, the value of one variable quantity, say y, depends on the value of another variable quantity, which we often call x. We say that "y is a function of x" and write this symbolically as

$$y = f(x) \qquad (\text{“}y \text{ equals } f \text{ of } x\text{”}).$$

The symbol f represents the function, the letter x is the **independent variable** representing the input value to f, and y is the **dependent variable** or output value of f at x.

> **DEFINITION** A **function** f from a set D to a set Y is a rule that assigns a *unique* value $f(x)$ in Y to each x in D.

The set D of all possible input values is called the **domain** of the function. The set of all output values of $f(x)$ as x varies throughout D is called the **range** of the function. The range might not include every element in the set Y. The domain and range of a function can be any sets of objects, but often in calculus they are sets of real numbers interpreted as points of a coordinate line. (In Chapters 12–15, we will encounter functions for which the elements of the sets are points in the plane, or in space.)

Often a function is given by a formula that describes how to calculate the output value from the input variable. For instance, the equation $A = \pi r^2$ is a rule that calculates the area A of a circle from its radius r. When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of real x-values for which the formula gives real y-values. This is called the **natural domain** of f. If we want to restrict the domain in some way, we must say so. The domain of $y = x^2$ is the entire set of real numbers. To restrict the domain of the function to, say, positive values of x, we would write "$y = x^2, x > 0$."

Changing the domain to which we apply a formula usually changes the range as well. The range of $y = x^2$ is $[0, \infty)$. The range of $y = x^2, x \geq 2$, is the set of all numbers obtained by squaring numbers greater than or equal to 2. In set notation (see Appendix 1), the range is $\{x^2 | x \geq 2\}$ or $\{y | y \geq 4\}$ or $[4, \infty)$.

When the range of a function is a set of real numbers, the function is said to be **real-valued**. The domains and ranges of most real-valued functions we consider are intervals or combinations of intervals. Sometimes the range of a function is not easy to find.

A function f is like a machine that produces an output value $f(x)$ in its range whenever we feed it an input value x from its domain (Figure 1.1). The function keys on a calculator give an example of a function as a machine. For instance, the $\sqrt{x}$ key on a calculator gives an output value (the square root) whenever you enter a nonnegative number x and press the $\sqrt{x}$ key.

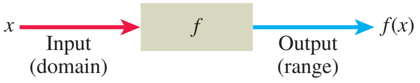


FIGURE 1.1 A diagram showing a function as a kind of machine.

A function can also be pictured as an **arrow diagram** (Figure 1.2). Each arrow associates to an element of the domain D a single element in the set Y. In Figure 1.2, the arrows indicate that $f(a)$ is associated with a, $f(x)$ is associated with x, and so on. Notice that a function can have the same *output value* for two different input elements in the domain (as occurs with $f(a)$ in Figure 1.2), but each input element x is assigned a *single* output value $f(x)$.

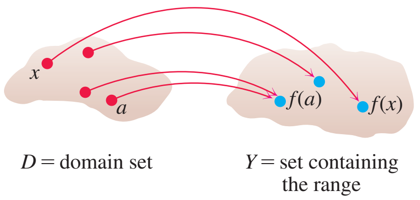


FIGURE 1.2 A function from a set D to a set Y assigns a unique element of Y to each element in D.

EXAMPLE 1 Verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of x for which the formula makes sense.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution The formula $y = x^2$ gives a real y-value for any real number x, so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root: $y = \left(\sqrt{y}\right)^2$ for $y \geq 0$.

The formula $y = 1/x$ gives a real y-value for every x except $x = 0$. For consistency in the rules of arithmetic, *we cannot divide any number by zero*. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$. That is, for $y \neq 0$ the number $x = 1/y$ is the input that is assigned to the output value y.

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).

In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives nonnegative real y-values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all nonnegative numbers.

The formula $y = \sqrt{1 - x^2}$ gives a real y-value for every x in the closed interval from -1 to 1. Outside this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1 - x^2}$ is $[0, 1]$. ■

Graphs of Functions

If f is a function with domain D, its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f. In set notation, the graph is

$$\{(x, f(x)) \mid x \in D\}.$$

The graph of the function $f(x) = x + 2$ is the set of points with coordinates (x, y) for which $y = x + 2$. Its graph is the straight line sketched in Figure 1.3.

The graph of a function f is a useful picture of its behavior. If (x, y) is a point on the graph, then $y = f(x)$ is the height of the graph above (or below) the point x. The height may be positive or negative, depending on the sign of $f(x)$ (Figure 1.4).

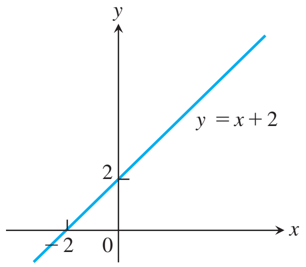


FIGURE 1.3 The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

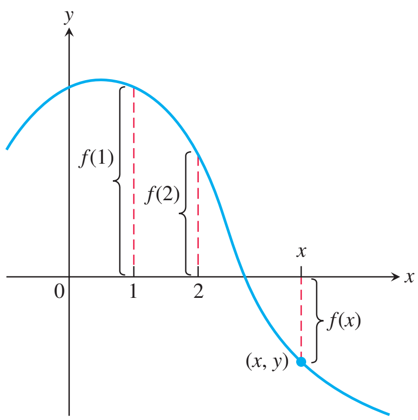


FIGURE 1.4 If (x, y) lies on the graph of f, then the value $y = f(x)$ is the height of the graph above the point x (or below x if $f(x)$ is negative).

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

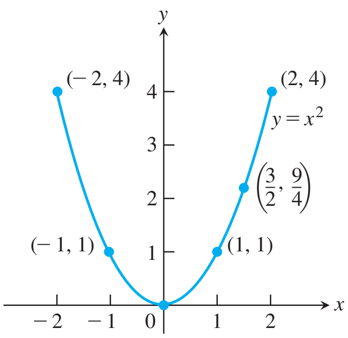
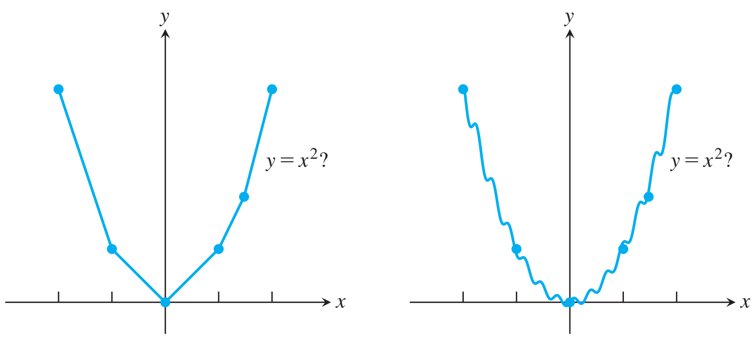


FIGURE 1.5 Graph of the function in Example 2.

EXAMPLE 2 Graph the function $y = x^2$ over the interval $[-2, 2]$.

Solution Make a table of xy-pairs that satisfy the equation $y = x^2$. Plot the points (x, y) whose coordinates appear in the table, and draw a *smooth* curve (labeled with its equation) through the plotted points (see Figure 1.5). ■

How do we know that the graph of $y = x^2$ doesn't look like one of these curves?



To find out, we could plot more points. But how would we then connect *them*? The basic question still remains: How do we know for sure what the graph looks like between the points we plot? Calculus answers this question, as we will see in Chapter 4. Meanwhile, we will have to settle for plotting points and connecting them as best we can.

Representing a Function Numerically

We have seen how a function may be represented algebraically by a formula and visually by a graph (Example 2). Another way to represent a function is **numerically**, through a table of values. Numerical representations are often used by engineers and experimental scientists. From an appropriate table of values, a graph of the function can be obtained using the method illustrated in Example 2, possibly with the aid of a computer. The graph consisting of only the points in the table is called a **scatterplot**.

EXAMPLE 3 Musical notes are pressure waves in the air. The data associated with Figure 1.6 give recorded pressure displacement versus time in seconds of a musical note produced by a tuning fork. The table provides a representation of the pressure function (in micropascals) over time. If we first make a scatterplot and then connect the data points (t, p) from the table, we obtain the graph shown in the figure.

Time	Pressure	Time	Pressure
0.00091	−0.080	0.00362	0.217
0.00108	0.200	0.00379	0.480
0.00125	0.480	0.00398	0.681
0.00144	0.693	0.00416	0.810
0.00162	0.816	0.00435	0.827
0.00180	0.844	0.00453	0.749
0.00198	0.771	0.00471	0.581
0.00216	0.603	0.00489	0.346
0.00234	0.368	0.00507	0.077
0.00253	0.099	0.00525	−0.164
0.00271	−0.141	0.00543	−0.320
0.00289	−0.309	0.00562	−0.354
0.00307	−0.348	0.00579	−0.248
0.00325	−0.248	0.00598	−0.035
0.00344	−0.041		

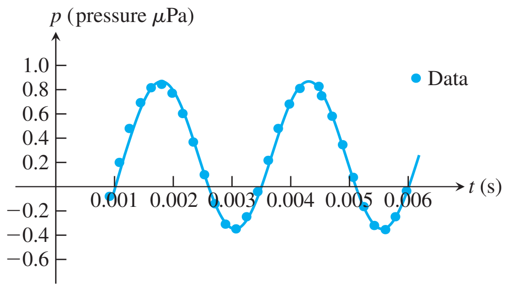


FIGURE 1.6 A smooth curve through the plotted points gives a graph of the pressure function represented by the accompanying tabled data (Example 3).

The Vertical Line Test for a Function

Not every curve in the coordinate plane can be the graph of a function. A function f can have only one value $f(x)$ for each x in its domain, so *no vertical* line can intersect the graph of a function more than once. If a is in the domain of the function f, then the vertical line $x = a$ will intersect the graph of f at the single point $(a, f(a))$.

A circle cannot be the graph of a function, since some vertical lines intersect the circle twice. The circle graphed in Figure 1.7a, however, contains the graphs of two functions of x, namely the upper semicircle defined by the function $f(x) = \sqrt{1 - x^2}$ and the lower semicircle defined by the function $g(x) = -\sqrt{1 - x^2}$ (Figures 1.7b and 1.7c).

Piecewise-Defined Functions

Sometimes a function is described in pieces by using different formulas on different parts of its domain. One example is the **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 \quad \text{First formula} \\ -x, & x < 0 \quad \text{Second formula} \end{cases}$$

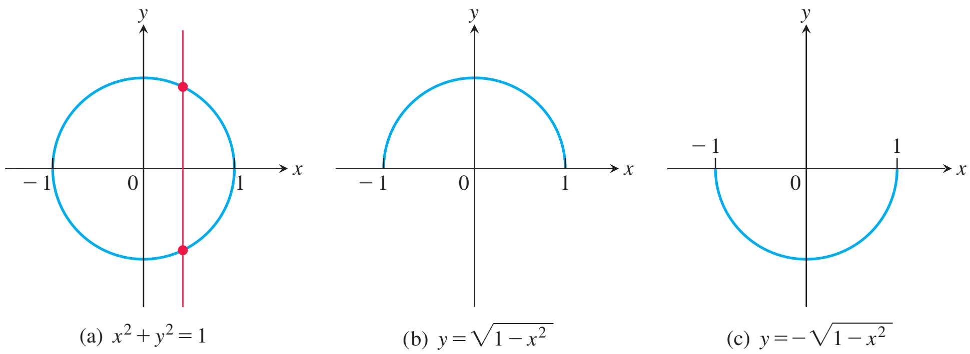


FIGURE 1.7 (a) The circle is not the graph of a function; it fails the vertical line test. (b) The upper semicircle is the graph of the function $f(x) = \sqrt{1 - x^2}$. (c) The lower semicircle is the graph of the function $g(x) = -\sqrt{1 - x^2}$.

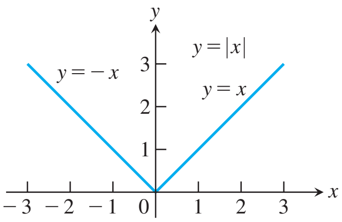


FIGURE 1.8 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

whose graph is given in Figure 1.8. The right-hand side of the equation means that the function equals x if $x \geq 0$, and equals $-x$ if $x < 0$. Piecewise-defined functions often arise when real-world data are modeled. Here are some other examples.

EXAMPLE 4 The function

$$f(x) = \begin{cases} -x, & x < 0 & \text{First formula} \\ x^2, & 0 \leq x \leq 1 & \text{Second formula} \\ 1, & x > 1 & \text{Third formula} \end{cases}$$

is defined on the entire real line but has values given by different formulas, depending on the position of x. The values of f are given by $y = -x$ when $x < 0$, $y = x^2$ when $0 \leq x \leq 1$, and $y = 1$ when $x > 1$. The function, however, is *just one function* whose domain is the entire set of real numbers (Figure 1.9). ■

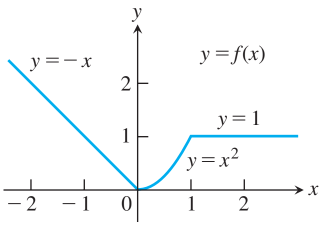


FIGURE 1.9 To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain (Example 4).

EXAMPLE 5 The function whose value at any number x is the *greatest integer less than or equal to* x is called the **greatest integer function** or the **integer floor function**. It is denoted $\lfloor x \rfloor$. Figure 1.10 shows the graph. Observe that

$$\lfloor 2.4 \rfloor = 2, \quad \lfloor 1.9 \rfloor = 1, \quad \lfloor 0 \rfloor = 0, \quad \lfloor -1.2 \rfloor = -2,$$
$$\lfloor 2 \rfloor = 2, \quad \lfloor 0.2 \rfloor = 0, \quad \lfloor -0.3 \rfloor = -1, \quad \lfloor -2 \rfloor = -2.$$ ■

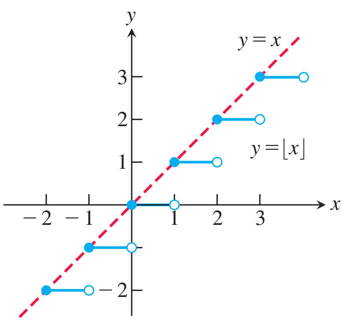


FIGURE 1.10 The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for x (Example 5).

EXAMPLE 6 The function whose value at any number x is the *smallest integer greater than or equal to* x is called the **least integer function** or the **integer ceiling function**. It is denoted $\lceil x \rceil$. Figure 1.11 shows the graph. For positive values of x, this function might represent, for example, the cost of parking x hours in a parking lot that charges $1 for each hour or part of an hour. ■

Increasing and Decreasing Functions

If the graph of a function climbs or rises as you move from left to right, we say that the function is *increasing*. If the graph descends or falls as you move from left to right, the function is *decreasing*.

DEFINITIONS Let f be a function defined on an interval I and let x_1 and x_2 be two distinct points in I.

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

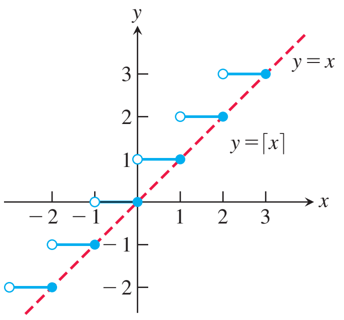


FIGURE 1.11 The graph of the least integer function $y = \lceil x \rceil$ lies on or above the line $y = x$, so it provides an integer ceiling for x (Example 6).

It is important to realize that the definitions of increasing and decreasing functions must be satisfied for *every* pair of points x_1 and x_2 in I with $x_1 < x_2$. Because we use the inequality $<$ to compare the function values, instead of $\leq$, it is sometimes said that f is *strictly* increasing or decreasing on I. The interval I may be finite (also called bounded) or infinite (unbounded).

EXAMPLE 7 The function graphed in Figure 1.9 is decreasing on $(-\infty, 0)$ and increasing on $(0, 1)$. The function is neither increasing nor decreasing on the interval $(1, \infty)$ because the function is constant on that interval, and hence the strict inequalities in the definition of increasing or decreasing are not satisfied on $(1, \infty)$. ■

Even Functions and Odd Functions: Symmetry

The graphs of *even* and *odd* functions have special symmetry properties.

DEFINITIONS A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,
odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

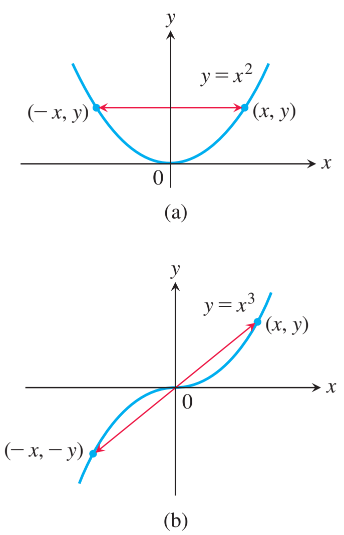


FIGURE 1.12 (a) The graph of $y = x^2$ (an even function) is symmetric about the y-axis. (b) The graph of $y = x^3$ (an odd function) is symmetric about the origin.

The names *even* and *odd* come from powers of x. If y is an even power of x, as in $y = x^2$ or $y = x^4$, it is an even function of x because $(-x)^2 = x^2$ and $(-x)^4 = x^4$. If y is an odd power of x, as in $y = x$ or $y = x^3$, it is an odd function of x because $(-x)^1 = -x$ and $(-x)^3 = -x^3$.

The graph of an even function is **symmetric about the y-axis**. Since $f(-x) = f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph (Figure 1.12a). A reflection across the y-axis leaves the graph unchanged.

The graph of an odd function is **symmetric about the origin**. Since $f(-x) = -f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph (Figure 1.12b). Equivalently, a graph is symmetric about the origin if a rotation of 180° about the origin leaves the graph unchanged. Notice that the definitions imply that both x and $-x$ must be in the domain of f.

EXAMPLE 8 Here are several functions illustrating the definitions.

$f(x) = x^2$ — Even function: $(-x)^2 = x^2$ for all x; symmetry about y-axis. So $f(-3) = 9 = f(3)$. Changing the sign of x does not change the value of an even function.

$f(x) = x^2 + 1$ — Even function: $(-x)^2 + 1 = x^2 + 1$ for all x; symmetry about y-axis (Figure 1.13a).

$f(x) = x$ — Odd function: $(-x) = -x$ for all x; symmetry about the origin. So $f(-3) = -3$ while $f(3) = 3$. Changing the sign of x changes the sign of an odd function.

$f(x) = x + 1$ — Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal.
Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure 1.13b). ■

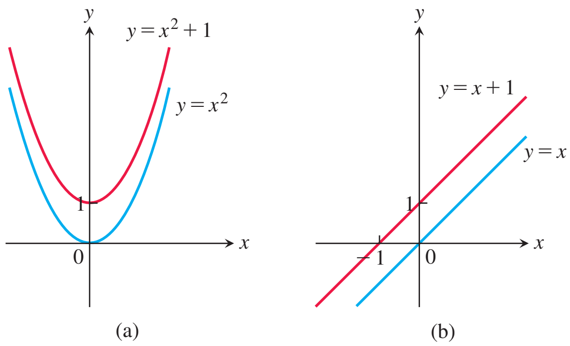


FIGURE 1.13 (a) When we add the constant term 1 to the function $y = x^2$, the resulting function $y = x^2 + 1$ is still even and its graph is still symmetric about the y-axis. (b) When we add the constant term 1 to the function $y = x$, the resulting function $y = x + 1$ is no longer odd, since the symmetry about the origin is lost. The function $y = x + 1$ is also not even (Example 8).

Common Functions

A variety of important types of functions are frequently encountered in calculus.

Linear Functions A function of the form $f(x) = mx + b$, where m and b are fixed constants, is called a **linear function**. Figure 1.14a shows an array of lines $f(x) = mx$. Each of these has $b = 0$, so these lines pass through the origin. The function $f(x) = x$ where $m = 1$ and $b = 0$ is called the **identity function**. Constant functions result when the slope is $m = 0$ (Figure 1.14b).

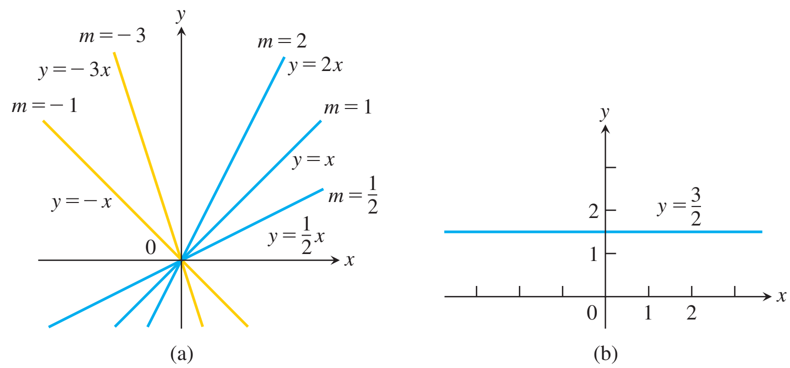


FIGURE 1.14 (a) Lines through the origin with slope m. (b) A constant function with slope $m = 0$.

> **DEFINITION** Two variables y and x are **proportional** (to one another) if one is always a constant multiple of the other—that is, if $y = kx$ for some nonzero constant k.

If the variable y is proportional to the reciprocal $1/x$, then sometimes it is said that y is **inversely proportional** to x (because $1/x$ is the multiplicative inverse of x).

Power Functions A function $f(x) = x^a$, where a is a constant, is called a **power function**. There are several important cases to consider.

(a) $f(x) = x^a$ with $a = n$, a positive integer.

The graphs of $f(x) = x^n$, for $n = 1, 2, 3, 4, 5$, are displayed in Figure 1.15. These functions are defined for all real values of x. Notice that as the power n gets larger, the curves tend to flatten toward the x-axis on the interval $(-1, 1)$ and to rise more steeply for $|x| > 1$. Each curve passes through the point $(1, 1)$ and through the origin. The graphs of functions with even powers are symmetric about the y-axis; those with odd powers are symmetric about the origin. The even-powered functions are decreasing on the interval $(-\infty, 0]$ and increasing on $[0, \infty)$; the odd-powered functions are increasing over the entire real line $(-\infty, \infty)$.

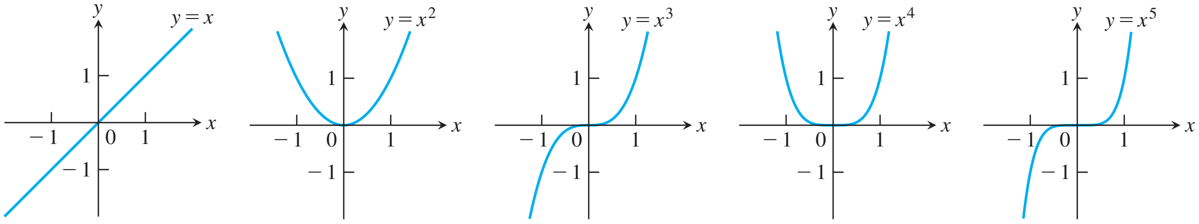


FIGURE 1.15 Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$, defined for $-\infty < x < \infty$.

(b) $f(x) = x^a$ with $a = -1$ or $a = -2$.

The graphs of the functions $f(x) = x^{-1} = 1/x$ and $g(x) = x^{-2} = 1/x^2$ are shown in Figure 1.16. Both functions are defined for all $x \neq 0$ (you can never divide by zero). The graph of $y = 1/x$ is the hyperbola $xy = 1$, which approaches the coordinate axes far from the origin. The graph of $y = 1/x^2$ also approaches the coordinate axes. The graph of the function f is symmetric about the origin; f is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$. The graph of the function g is symmetric about the y-axis; g is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

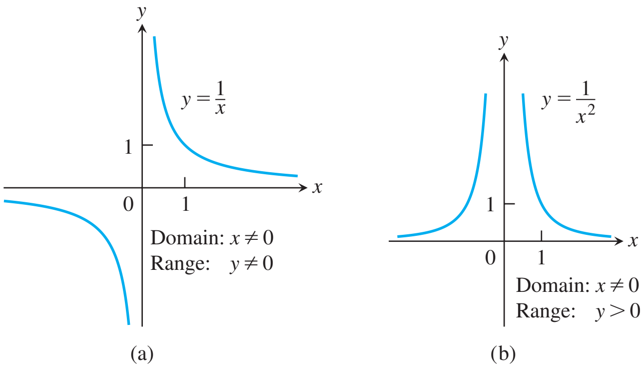


FIGURE 1.16 Graphs of the power functions $f(x) = x^a$. (a) $a = -1$, (b) $a = -2$.

(c) $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt[3]{x}$ are the **square root** and **cube root** functions, respectively. The domain of the square root function is $[0, \infty)$, but the cube root function is defined for all real x. Their graphs are displayed in Figure 1.17, along with the graphs of $y = x^{3/2}$ and $y = x^{2/3}$. (Recall that $x^{3/2} = (x^{1/2})^3$ and $x^{2/3} = (x^{1/3})^2$.)

Polynomials A function p is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are real constants (called the **coefficients** of the polynomial). All polynomials have domain $(-\infty, \infty)$. If the

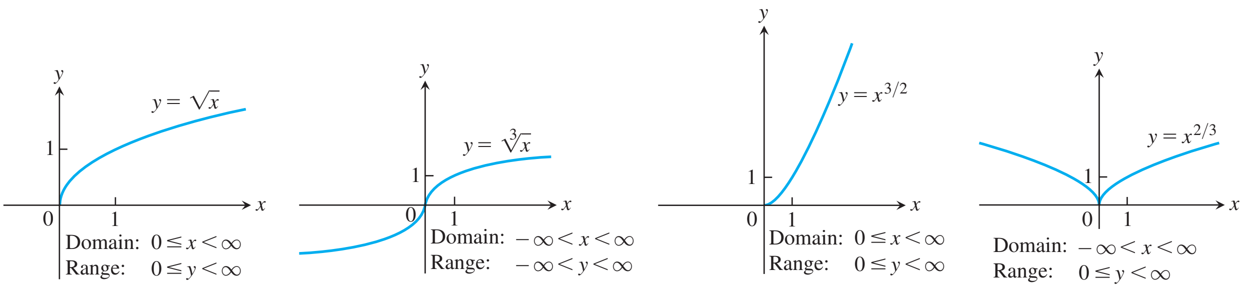


FIGURE 1.17 Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

leading coefficient $a_n \neq 0$, then n is called the **degree** of the polynomial. Linear functions with $m \neq 0$ are polynomials of degree 1. Polynomials of degree 2, usually written as $p(x) = ax^2 + bx + c$, are called **quadratic functions**. Likewise, **cubic functions** are polynomials $p(x) = ax^3 + bx^2 + cx + d$ of degree 3. Figure 1.18 shows the graphs of three polynomials. Techniques to graph polynomials are studied in Chapter 4.

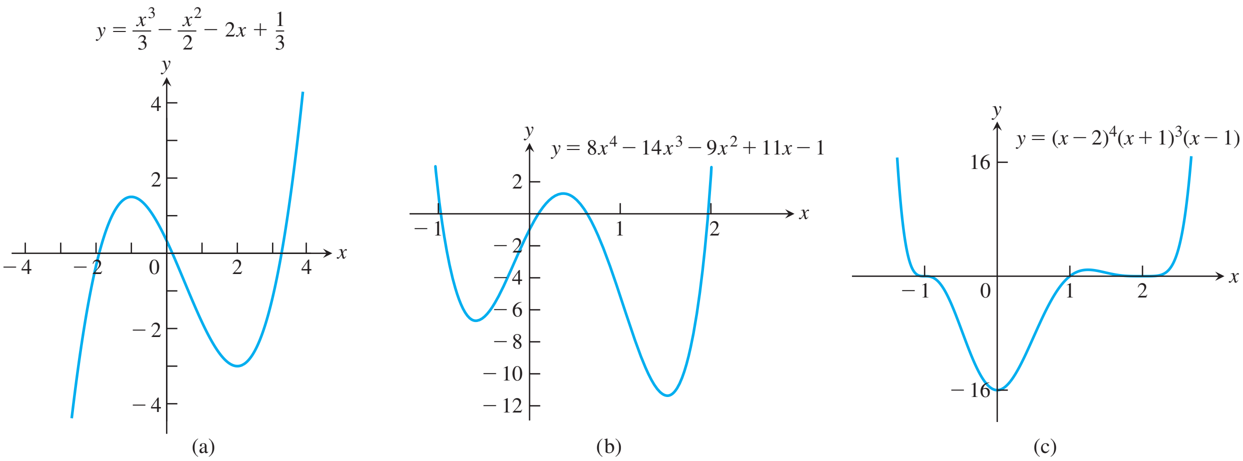


FIGURE 1.18 Graphs of three polynomial functions.

Rational Functions A **rational function** is a quotient or ratio $f(x) = p(x)/q(x)$, where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$. The graphs of several rational functions are shown in Figure 1.19.

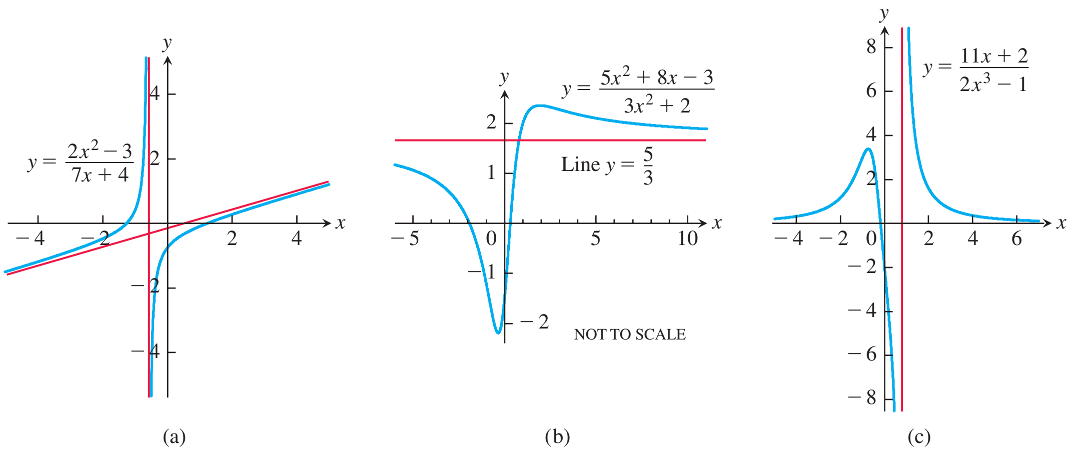


FIGURE 1.19 Graphs of three rational functions. The straight red lines approached by the graphs are called *asymptotes* and are not part of the graphs. We discuss asymptotes in Section 2.5.

Algebraic Functions Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots) lies within the class of **algebraic functions**. All rational functions are algebraic, but also included are more complicated functions (such as those satisfying an equation like $y^3 - 9xy + x^3 = 0$, studied in Section 3.7). Figure 1.20 displays the graphs of three algebraic functions.

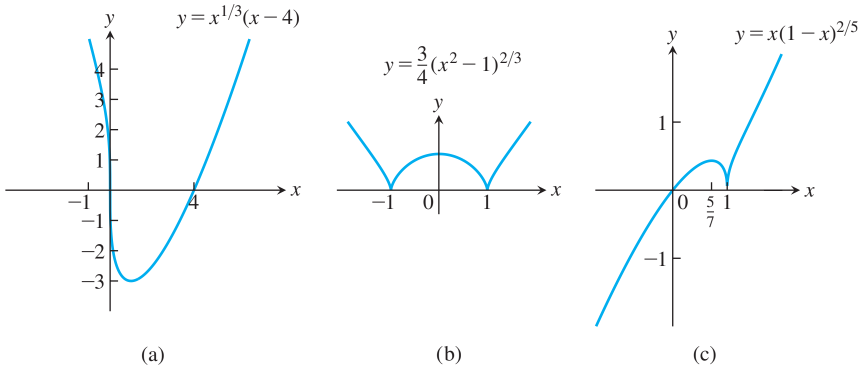


FIGURE 1.20 Graphs of three algebraic functions.

Trigonometric Functions The six basic trigonometric functions are reviewed in Section 1.3. The graphs of the sine and cosine functions are shown in Figure 1.21.

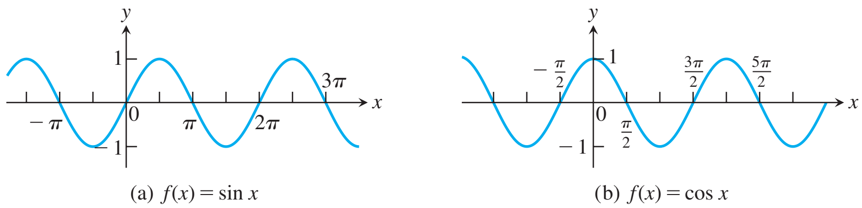


FIGURE 1.21 Graphs of the sine and cosine functions.

Exponential Functions A function of the form $f(x) = a^x$, where $a > 0$ and $a \neq 1$, is called an **exponential function** (with base a). All exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$, so an exponential function never assumes the value 0. We discuss exponential functions in Section 1.4. The graphs of some exponential functions are shown in Figure 1.22.

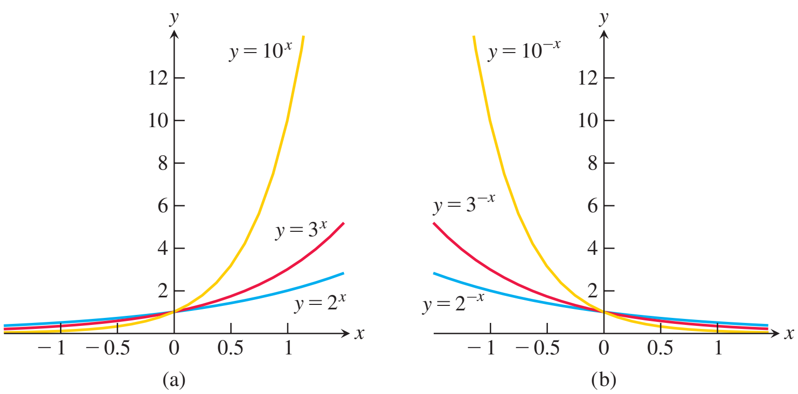


FIGURE 1.22 Graphs of exponential functions.